Upper and lower bounds for 3-dimensional k-within-consecutive-(r_1, r_2, r_3)-out-of-(n_1, n_2, n_3):F system

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Abstract

As a 2-dimensional k-within-consecutive-r-out-of-n:F system, for example, there are connected-(r, s)-out-of-(m, n):F lattice system and 2-dimensional k-within-consecutive-(r, s)-out-of-(m, n):F system. For these systems, the calculation method for reliability and, upper and lower bounds, have been studied by many researchers. Furthermore, several researches had dealt with the reliability of more multi-dimensional systems. In this study, we consider 3-dimensional k-within-consecutive-r-out-of-n:F system, called the 3-dimensional k-within-consecutive-(r_1, r_2, r_3)-out-of-(n_1, n_2, n_3):F system. In this system, although an enumeration method could be used for evaluating the exact system reliability of very small-sized systems, the method takes much computing time when applied to larger systems. Therefore, developing upper and lower bounds is useful for calculating the reliability of large systems in a reasonable execution time. In this study, we propose the upper and lower bounds for reliabilities of a 3-dimensional k-within-consecutive-(r_1, r_2, r_3)-out-of-(n_1, n_2, n_3):F system, by enhancing the basic idea applied to the methods for estimating the reliability of a 2-dimensional k-within-consecutive-(r, s)-out-of-(m, n):F system.

1. Introduction

The consecutive-k-out-of-n:F systems have been extensively studied since the early 1980s. This type of system can be regarded as a one-dimensional reliability model and can be extended to 2- or 3- or d-dimensional versions ($d \geq 2$). There are a few papers treating 3-dimensional systems, whose reliability is equal to the probability that a radiologist might not detect their presence of a disease by Salvia and Lasher(1990). We have not yet obtained the efficient algorithm to estimate reliability of such a complex system. For a 3-dimensional k-within-consecutive-(r_1, r_2, r_3)-out-of-(n_1, n_2, n_3):F system (denoted as k/(r_1, r_2, r_3) /(n_1, n_2, n_3):F system throughout this paper), which is a 3-dimensional version of consecutive-k-within-r-out-of-n:F system. This system consists of $n_1 \times n_2 \times n_3$ components, which are arranged like a (n_1, n_2, n_3) rectangular solid. This system fails if and only if there is a (r_1, r_2, r_3) rectangular solid in which k or more components fails as shown in Figure 1. In this study, a component (h, i, j) means the component located on h-th point in the n_1 axis, i-th point in the n_2 axis and j-th point in the n_3 axis, with reliability p_{hij} and failure probability $q_{hij} = 1 - p_{hij}$, for $h = 1,2,\cdots,n_1$, $i = 1,2,\cdots,n_2$ and $j = 1,2,\cdots,n_3$, as shown in Figure 1. Salvia and Lasher(1990) *etc.* gave the following examples to illustrate where such multi-dimensional models may be used, the presence of a disease is diagnosed by reading an X-ray. The other examples, k/(r_1, r_2, r_3)/(n_1, n_2, n_3):F system can be applied to the mathematical model of a 3-dimensional flash memory cell failure model, and hypercube topology of the connection network, and so on. In this system, although an enumeration method could be used for evaluating the exact system reliability of very small-sized systems, the method takes much computing time when applied to larger systems. Therefore, developing upper and lower bounds is useful for evaluating the reliability of large systems in a reasonable computing time. In this study, we propose the upper and lower bounds for reliabilities of a k/(r_1, r_2, r_3) /(n_1, n_2, n_3):F system, by enhancing the basic idea applied to the methods for estimating the reliability of a 2-dimensional k-within-consecutive-(r, s)-out-of-(m, n):F system (Yamamoto and Akiba (submitted)).

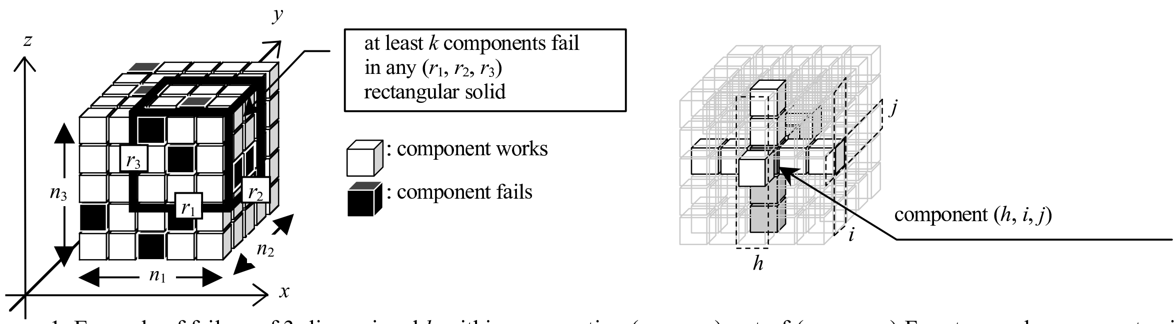

Figure 1: Example of failure of 3-dimensional k-within-consecutive-(r_1, r_2, r_3)-out-of-(n_1, n_2, n_3):F system and component axis

2. Upper and Lower bounds for the reliability

In this section, we propose upper and lower bounds for the reliability of a $k/(r_1, r_2, r_3)/(n_1, n_2, n_3)$:F system. For this, we introduce some notations. For the simple expression of theorems and equations, the virtual components should be stated: component (h, i, j) with component reliability 1.0, for $(h, i, j) \in \{(h, i, j) \mid 1 \le h \le n_1, 1 \le i \le n_2, 1 \le j \le n_3\}^c$. Furthermore, we denote some events, which occur on the above sets. For $h = 1,2,\cdots,n_1$, $i = 1,2,\cdots,n_2$ and $j = 1,2,\cdots,n_3$, we define a event as follows.

S_{hij} : The event that "k or more components fail in a (r_1, r_2, r_3) rectangular solid with components (h, i, j) as its upper right deep apex" and "at least one component fails in a (r_1, r_2) matrix with components $(h–r_1+1, i–r_2+1, j)$, $(h, i–r_2+1, j)$, $(h–r_1+1, i, j)$ and (h, i, j) as its apices" and "at least one component fails in a (r_1, r_3) matrix with components $(h–r_1+1, i, j–r_3+1)$, $(h, i, j–r_3+1)$, $(h–r_1+1, i, j)$ and (h, i, j) as its apices" and "at least one component fails in a (r_2, r_3) matrix with components $(h, i–r_2+1, j–r_3+1)$, $(h, i, j–r_3+1)$, $(h, i–r_2+1, j)$ and (h, i, j) as its apices".

For $h = r_1, r_1+1,\cdots,n_1$, $i = r_2, r_2+1,\cdots,n_2$ and $j = r_3, r_3+1,\cdots,n_3$, we define some events as follows. Now, we define some sets of components in the system for $h = 1,2,\cdots,n_1$, $i = 1,2,\cdots,n_2$ and $j = 1,2,\cdots,n_3$ $CG_1(h, i, j)$ is a set of all components in a $(r_1–1, r_2–1, r_3–1)$ rectangular solid with components $(h–1, i–1, j–1)$ as its upper right deep apex, $CG_2(h, i, j)$ is a set of all components in a $(r_2–1, r_3–1)$ matrix with components $(h, i–1, j–1)$, $(h, i–r_2+1, j–1)$, $(h, i–r_2+1, j–r_3+1)$ and $(h, i–1, j–r_3+1)$ as its apices, $CG_3(h, i, j)$ is a set of all components in a $(r_1–1, r_3–1)$ matrix with components $(h–1, i, j–1)$, $(h–r_1+1, i, j–1)$, $(h–r_1+1, i, j–r_3+1)$ and $(h–1, i, j–r_3+1)$ as its apices, $CG_4(h, i, j)$ is a set of all components in a $(r_1–1, r_2–1)$ matrix with components $(h–1, i–1, j)$, $(h–r_1+1, i–1, j)$, $(h–r_1+1, i–r_2+1, j)$ and $(h–1, i–r_2+1, j)$ as its apices, $CG_5(h, i, j)$ is a set of $r_2–1$ components with components $(h, i–1, j)$, $(h, i–2, j)$,…,$(h, i–r_2+1, j)$, $CG_6(h, i, j)$ is a set of $r_3–1$ components with components $(h, i, j–1)$, $(h, i, j–2)$,…, $(h, i, j–r_3+1)$, $CG_7(h, i, j)$ is a set of $r_1–1$ components with components $(h–1, i, j)$, $(h–2, i, j)$,…, $(h–r_1+1, i, j)$.

C_{hij} : The event that all components function in set of all components in a solid with components $(h–2r_1+2, i, j)$, $(h–r_1, i, j)$, $(h–r_1, i–r_2, j)$, $(h+1, i–r_2, j)$, $(h+1, i–1, j)$, $(h+1, i, j–1)$, $(h+r_1–1, i, j–1)$, $(h+r_1–1, i, j–r_3+1)$, $(h–2r_1+2, i, j–r_3+1)$, $(h–2r_1+2, i–r_2+2, j–r_3+1)$, $(h+r_1–1, i–r_2+2, j–r_3+1)$, $(h–2r_1+2, i–r_2+2, j)$ and $(h+r_1–1, i–r_2+2, j)$ as its apices.

G_{hij} : The whole event for $h = r_1$, $i = r_2$, $j = r_3$. The event that less than k components fail in $CG_1(h,i,j) \cap CG_3(h,i,j) \cup CG_4(h,i,j) \cup CG_7(h,i,j)$ for $h \ne r_1, i = r_2, j = r_3$. For $h = r_1, i \ne r_2, j = r_3$ and $h = r_1, i = r_2, j \ne r_3$, G_{hij} is similar events for $h \ne r_1, i = r_2, j = r_3$. The event that "less than k components fail in $CG_1(h,i,j) \cap CG_2(h,i,j) \cup CG_4(h,i,j) \cup CG_5(h,i,j)$" and "less than k components fail in $CG_1(h,i,j) \cap CG_2(h,i,j) \cup CG_3(h,i,j) \cup CG_6(h,i,j)$" for $h = r_1, i \ne r_2, j \ne r_3$. For $h \ne r_1, i = r_2, j \ne r_3$ and $h \ne r_1, i \ne r_2, j = r_3$, G_{hij} is similar events for $h = r_1, i \ne r_2, j \ne r_3$. The event that "less than k components fail in $CG_1(h,i,j) \cap CG_2(h,i,j) \cup CG_4(h,i,j) \cup CG_5(h,i,j)$" and "less than k components fail in $CG_1(h,i,j) \cap CG_2(h,i,j) \cup CG_3(h,i,j) \cup CG_6(h,i,j)$" and "less than k components fail in $CG_1(h,i,j) \cap CG_3(h,i,j) \cup CG_4(h,i,j) \cup CG_7(h,i,j)$" for $h \ne r_1, i \ne r_2, j \ne r_3$,

E_{hij} : The event that "k or more components fail in a (r_1, r_2, r_3) rectangular solid with components (h, i, j) as its upper right deep apex" and "event G_{hij} occurs".

By using the above notations, our proposed upper and lower bounds for the reliability of a k/(r_1, r_2, r_3)/(n_1, n_2, n_3):F system are given in Theorem.

THEOREM: Upper bound UB and lower bound LB for the reliability of a k/(r_1, r_2, r_3)/(n_1, n_2, n_3):F system are given as follows.

$$LB = \prod_{h=1}^{n_1}\prod_{i=1}^{n_2}\prod_{j=1}^{n_3}[1-\Pr\{S_{hij}\}], \tag{1}$$

$$UB = \prod_{h=r_1}^{n_1}\prod_{i=r_2}^{n_2}\prod_{j=r_3}^{n_3}[1-\Pr\{C_{hij}\}\frac{\Pr\{E_{hij}\}}{\Pr\{G_{hij}\}}]. \tag{2}$$

In the i.i.d. cases, Corollary gives the upper and lower bounds with no description of S_{hij}, G_{hij} and E_{hij}.

COROLLARY: Let p be a component reliability.

(1) Lower bound LB_p is given as

$$LB_p = \prod_{h=1}^{n_1}\prod_{i=1}^{n_2}\prod_{j=1}^{n_3}\left[1-\sum_{t=k}^{uvw} N_L(t;h,i,j)(1-p)^t p^{uv-t}\right], \tag{4}$$

where

$$N_L(t;h,i,j) = \binom{uvw}{t} - \binom{(u-1)vw}{t} - \binom{u(v-1)w}{t} - \binom{uv(w-1)}{t} + \binom{u(v-1)(w-1)}{t} + \binom{(u-1)v(w-1)}{t}$$
$$+\binom{(u-1)(v-1)w}{t} - \binom{(u-1)(v-1)(w-1)}{t}, \tag{5}$$

$$u = \min\{h,r_1\}, \quad v = \min\{i,r_2\}, \quad w = \min\{j,r_3\}. \tag{6}$$

(2) Upper bound UB_p is given as

$$UB_p = \left[1-\sum_{t=k}^{r_1r_2r_3}\binom{r_1r_2r_3}{t}(1-p)^t p^{r_1r_2r_3-t}\right] \prod_{\substack{r_1\le h\le n_1 \\ r_2\le i\le n_2 \\ r_3\le j\le n_3 \\ (h,i,j)\ne(r_1,r_2,r_3)}} \left[1-p^{\#C(h,i,j)}\frac{\sum_{t=k}^{r_1r_2r_3} N_E(t;h,i,j)(1-p)^t p^{r_1r_2r_3-t}}{\sum_{t=0}^{\#G(h,i,j)} N_G(t;h,i,j)(1-p)^t p^{\#G(h,i,j)-t}}\right], \tag{7}$$

where for $h \ne r_1,\ i \ne r_2,\ j \ne r_3$,

$$\#G(h,i,j) = \begin{cases} r_1r_2(r_3-1) & (h=r_1, i=r_2, j\ne r_3), \\ r(r_2-1)r_3 & (h=r_1, i\ne r_2, j=r_3), \\ (r_1-1)r_2r_3 & (h\ne r_1, i=r_2, j=r_3), \\ r_1r_2r_3-r_1 & (h=r_1, i\ne r_2, j\ne r_3), \\ r_1r_2r_3-r_2 & (h\ne r_1, i=r_2, j\ne r_3), \\ r_1r_2r_3-r_3 & (h\ne r_1, i\ne r_2, j=r_3), \\ r_1r_2r_3-1 & \text{otherwise}, \end{cases} \tag{8}$$

$$\#C(h,\ i,\ j) = l_a(l_c+r_2)(l_d+r_3) + l_b((l_c+r_2)(l_d+r_3)-1) + r_1(l_c\ (l_d+r_3) + l_d r_2, \tag{9}$$

l_a=max{0, min{h–r_1, r_1–1}}, l_b=min{r_1–1, n_1–h}, l_c=max{0, min{i–r_2, r_2–1}}, l_d=max{0, min{j–r_3, r_3–1}}.

Now, $N_G(t;h,i,j)$ and $N_E(t;h,i,j)$ are given as follows. For $h > r_1, i > r_2\ , j > r_3$,

Table 1: Upper and lower bounds for the reliability of $k/(r_1, r_2, r_3)/(n_1, n_2, n_3)$:F system

								Upper and lower bounds	
n_1	n_2	n_3	r_1	r_2	r_3	k	p	*UB*	*LB*
10	10	10	2	2	2	2	0.99000	0.435143	0.359604
10	10	10	2	2	2	2	0.99900	0.989817	0.989604
10	10	10	2	2	2	3	0.98000	0.855888	0.781153
10	10	10	2	2	2	3	0.99000	0.974983	0.968562
50	50	50	2	2	2	2	0.99900	0.219270	0.211320
50	50	50	2	2	2	2	0.99990	0.984578	0.984541
50	50	50	2	2	2	3	0.99500	0.571939	0.527560
50	50	50	2	2	2	3	0.99900	0.994963	0.994826
100	100	100	2	2	2	2	0.99980	0.602586	0.601092
100	100	100	2	2	2	2	0.99990	0.880757	0.880483
100	100	100	2	2	2	2	0.99999	0.998728	0.998728
100	100	100	2	2	2	3	0.99900	0.959341	0.958235
100	100	100	2	2	2	3	0.99910	0.970102	0.969367

$$N_G(t;h,i,j) = \begin{cases} \binom{r_1 r_2 r_3 - 1}{t} & (t \le k-1), \\ \displaystyle\sum_{\substack{x_1+x_2+x_3+x_4+x_5+x_6+x_7=t \\ x_1+x_2+x_4+x_5 \le k-1 \\ x_1+x_2+x_3+x_6 \le k-1 \\ x_1+x_3+x_4+x_7 \le k-1}} \binom{(r_1-1)(r_2-1)(r_3-1)}{x_1}\binom{(r_2-1)(r_3-1)}{x_2}\binom{(r_1-1)(r_3-1)}{x_3}\binom{(r_1-1)(r_2-1)}{x_4}\binom{(r_2-1)}{x_5}\binom{(r_3-1)}{x_6}\binom{(r_1-1)}{x_7} & (k \le t \le 3(k-1)), \\ 0 & (3(k-1) < t). \end{cases} \tag{10}$$

$$N_E(t;h,i,j) = N_G(t;h,i,j) + N_G(t-1;h,i,j). \tag{11}$$

Except the range of $h > r_1, i > r_2, j > r_3$, $N_G(t;h,i,j)$ and $N_E(t;h,i,j)$ are obtained similarly.

We calculated upper and lower bounds for the reliability of a $k/(r_1, r_2, r_3)/(n_1, n_2, n_3)$:F systems with the identical component reliability. In Table 1, we show the some results of numerical experiments. For the system sizes, each of n_1, n_2 and n_3 takes the values of 10, 50 and 100. As the sizes of the rectangular solid which leads to each of r_1, r_2 and r_3 takes the value of 2. And, the number of failure components k takes the value of 2 and 3. From Table 1, we found the following within the range of our experiments, the difference between lower bound *LB* and upper bound *UB* becomes small when a system is large and component reliabilities are close to one.

3. Conclusion

In this study, we propose the upper and lower bounds for reliabilities of a 3-dimensional k-within-consecutive-(r_1, r_2, r_3)-out-of-(n_1, n_2, n_3):F system. Consequently, we found the following characteristic of the system through the experimental results: the difference between our proposed lower bound and upper bound becomes small when the size of a system is large and components reliabilities are close to one.

References

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